

Mass correction and gravitational energy radiation in black hole perturbation theory

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Abstract

Using second order black hole perturbation theory, we show that the difference between the ADM mass and the final black hole mass, computed to the lowest significant order, is equal, to the same order, to the total gravitational radiation energy, obtained applying the Landau and Lifschitz (pseudotensor) equation to the first order perturbation. This result may be considered as a consistency check for the theory.

PACS numbers: 04.25.Nx, 04.70.Bw

1 Introduction

The analysis of the motion of gravitating bodies in the strong field region is, clearly, one of the most important problems in General Relativity, since it is here where one would expect to find the most pronounced effects, allowing, in principle, for a clear confrontation of the theory with observation. Perhaps the most important of these

effects, and the one to which the largest theoretical and experimental efforts is being devoted nowadays, is the emission of gravitational radiation through gravitational waves. As is well known, however, and in contrast with the situation in other theories, e.g. electromagnetism, the whole subject of gravitational waves and gravitational radiation in the context of general relativity is a very complex one, both from the point of view of their physical interpretation as from that of the difficulties in constructing solutions, (either exact or sufficiently approximate), appropriate for the modeling of relevant astrophysical systems.

From very general arguments, one expects that the most prominent sources of bursts of gravitational radiation, are the coalescences of two astrophysical objects, leading to a single black hole as the final result. These same general ideas indicate that most of the radiation, and therefore of the information to be observed at a large distance from the region of emission, should come from the last stages of this coalescence, where the system is close to, and rapidly approaching the final black hole stage. This has lead, among other lines of attack, to the idea that, given adequate initial data, one could consider these last stages as corresponding to the evolution of the perturbations of a suitable chosen black hole.

A successful application of this idea, attributed to Smarr, was carried out by Price and Pullin [1], in the case of the head-on collision of two black holes, in the close approximation, applying a technique, originally developed by Regge and Wheeler [2], and Zerilli [3], (see also [4]), devised precisely for the analysis of linearized perturbations of black hole. By successful we mean here that the results given perturbatively in [1], are essentially identical to those obtained by “exact” numerical solution of Einstein’s equations, for the same initial data, for a suitably chosen range of the parameters characterizing the problem.

It should be clear, however, that if the numerical results had not been available, it would have been essentially impossible to asses the range of validity of the perturbative results, not only because it relays only on linearized theory, but also because of the intuitively appealing, but non rigorous nature of the identification of what is meant by

the “gravitational energy” radiated by the system.

An important step in the direction of defining a range of validity for the perturbative treatment was given in [5], where the formalism was extended to second order, with the idea that second order corrections should serve as a sort of “error bars” on the first order computations. This formalism was applied to the head-on collision of black holes in [6], showing a very remarkable agreement between the “exact” result, the first order results and the associated “error bars”.

A somewhat different application of black hole perturbation theory, taken to second order, would be to analyze, instead of the gravitational radiation, the change in the mass parameter of the black hole. From a physical point of view, we expect that if an isolated astrophysical system contains, at a given time, certain mass-energy, and that if, after a long time, part of this mass-energy is radiated to infinity, the sum of the mass-energy remaining in the system and that radiated to infinity, should be equal to initial mass. In more rigorous terms, the initial mass-energy, for an isolated system, is given by the ADM mass. Since the process which we are envisioning is one where the final result is a black hole, the final mass-energy is simply that of the final black hole. The energy radiated to infinity should then be equal to the difference between these two masses. This comparison would provide an independent check of the validity of the computation of the radiated energy, as given in, e.g., [1].

In this note we consider the problem of computing the correction to the mass parameter in the lowest non trivial order in black hole perturbation theory. After reviewing briefly the formalism, we derive a general equation for the relevant metric perturbation function, and, under some general restrictions, and using the Zerilli equation, obtain the expression for the mass correction, in terms of the Zerilli function. It is remarkable that the resulting expression is identical to that obtained from the Landau-Lifschitz pseudotensor for the total gravitational energy radiation, as given in [4], although this is not mentioned anywhere in the derivations. In some sense, we might even consider this reassuring proof of consistency of the perturbation expansion, as a “derivation” of the radiation formula, since, as will be seen below, only general properties of the solutions

of Einstein's equations are used in the proofs, without any reference to asymptotically flat or radiation gauges. We should, however, make clear that it is not the intention of this article to delve into the difficult problem of the definition of the energy in general relativity, and of the relative merits of the different “pseudotensors” or “complexes” that have been discussed in the literature [9]. In particular, the comparison is made only to the expression for the integrated power, and it is known that different prescriptions may give here the same final answer. We also remark that the equivalence between the Bondi energy flux and that given by the Landau-Lifschitz complex, in suitable asymptotically flat space-times, has been shown in [10].

2 The perturbative expansion

The essence of the perturbation method is the expansion of the metric in the form

$$g_{\mu\nu}(\epsilon) = g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)} + \epsilon^2 g_{\mu\nu}^{(2)} + \mathcal{O}(\epsilon^3) \quad (1)$$

where $g_{\mu\nu}(\epsilon)$ represents a one parameter family of solutions, and $g_{\mu\nu}^{(0)}$ some known exact solution of Einstein's equations, leading to an expansion of the Einstein equation into a hierarchy of equations, ordered also by the parameter ϵ , where the solution of each order, considered as an initial value problem, requires the solution of all previous ones [5]. Some relevant issues, including gauge invariance, are discussed in more detail in [7]. (See also [8].)

In our case, $g_{\mu\nu}^{(0)}$ corresponds to the Schwarzschild black hole, and ϵ to some parameter characterizing the departure of the initial data from that of a black hole. A particularly suitable framework for this problem is given by the Regge-Wheeler [2] formalism, where the perturbations to any given order are given as a multipolar expansion in the angular variables θ , and ϕ . The multipoles are, in turn, separated into even and odd type, for any given L , the order of the multipole. The $L = 0$, (monopole), terms contain information on the mass of the system. We shall assume that the leading (first order in ϵ) part of the perturbation corresponds to the $L = 2$, even, (quadrupole) terms. (This is, for instance, the case considered in [1]). Therefore, all other terms are, at least, of order ϵ^2

We choose from the outset a Regge - Wheeler gauge [2]. This means that the non vanishing metric coefficients are

$$\begin{aligned}
g_{tt} &= -\left(1 - \frac{2m}{r}\right) \left[1 - \epsilon H_0^{(1)}(t, r) P_2(\theta) - \epsilon^2 H_0^{(2)} P_0\right] \\
g_{tr} &= \epsilon H_1^{(1)}(t, r) P_2(\theta) \\
g_{rr} &= \left(1 - \frac{2m}{r}\right)^{-1} \left[1 + \epsilon H_2^{(1)}(t, r) P_2(\theta) + \epsilon^2 H_2^{(2)} P_0\right] \\
g_{\theta\theta} &= r^2 [1 + \epsilon K^{(1)}(t, r) P_2(\theta)] \\
g_{\phi\phi} &= r^2 \sin(\theta)^2 [1 + \epsilon K^{(1)}(t, r) P_2(\theta)]
\end{aligned} \tag{2}$$

where the P_L , $L = 0, 2$ are Legendre polynomial in $\cos(\theta)$. The factor ϵ makes explicit the perturbation order. The functions $H_0^{(1)}(t, r)$, $H_1^{(1)}(t, r)$, $H_2^{(1)}(t, r)$, and $K^{(1)}(t, r)$ characterize the first order, $L = 2$, even, perturbations, while, $H_0^{(2)}(t, r)$, and $H_2^{(2)}(t, r)$ describe the $L = 0$, second order perturbations. The general second order expansion contains terms also for $L = 2$ and $L = 4$, but these are decoupled, to this order, from each other, and from the $L = 0$ terms, and we shall not consider them here.

We remark that although, in general, the $L = 0$ perturbations would also contain terms of the form

$$\begin{aligned}
g_{tr} &= \epsilon^2 H_1^{(2)}(t, r) P_0(\theta) \\
g_{\theta\theta} &= r^2 \epsilon^2 K^{(2)}(t, r) P_0(\theta) \\
g_{\phi\phi} &= r^2 \sin(\theta)^2 \epsilon^2 K^{(2)}(t, r) P_0(\theta)
\end{aligned} \tag{3}$$

it is always possible to choose a gauge where $H_1^{(2)}(t, r) = 0$, and $K^{(2)}(t, r) = 0$, so we make that simplifying choice. Furthermore, with this choice, if we assume that the first order perturbations vanish, the solution of Einstein's equations for $H_2^{(2)}$ is time independent, and of the form

$$H_2^{(2)} = \frac{C}{r - 2M} \tag{4}$$

where C is a constant.

Now, if we consider as the only "perturbation" a change in mass δM , it can be seen that

$$H_2^{(2)} = \frac{2\delta M}{r - 2M} \tag{5}$$

Therefore, we identify the constant C with twice the correction to the mass.

Taking into account the different powers in ϵ , and in a manner analogous to that in [5], the Einstein equations for (2) separate into a linear set of equations for $H_0^{(1)}(t, r)$, $H_1^{(1)}(t, r)$, $H_2^{(1)}(t, r)$, and $K^{(1)}(t, r)$, corresponding to terms linear in ϵ , and a set of equations linear in $H_0^{(2)}(t, r)$, and $H_2^{(2)}(t, r)$, but containing “source”-like terms, quadratic in the first order functions.

Regarding the first order functions, we recall that the general solution to the first order equations can be given in terms of the Zerilli function. Namely, any set $H_0^{(1)}(t, r)$, $H_1^{(1)}(t, r)$, $H_2^{(1)}(t, r)$, $K^{(1)}(t, r)$ of solutions of these equations can be written in the form

$$K^{(1)}(t, r) = 6 \frac{r^2 + rM + M^2}{r^2(2r + 3M)} \psi^{(1)}(t, r) + \left(1 - 2\frac{M}{r}\right) \frac{\partial \psi^{(1)}(t, r)}{\partial r} \quad (6)$$

$$H_2^{(1)}(t, r) = \frac{\partial}{\partial r} \left[\frac{2r^2 - 6rM - 3M^2}{r(2r + 3M)} \psi^{(1)}(t, r) + (r - 2M) \frac{\partial \psi^{(1)}(t, r)}{\partial r} \right] - K^{(1)}(t, r) \quad (7)$$

$$H_1^{(1)}(t, r) = \frac{2r^2 - 6rM - 3M^2}{(r - 2M)(2r + 3M)} \frac{\partial \psi^{(1)}(t, r)}{\partial t} + r \frac{\partial^2 \psi^{(1)}(t, r)}{\partial r \partial t} \quad (8)$$

$$H_0^{(1)}(t, r) = H_2^{(1)}(t, r) \quad (9)$$

where $\psi^{(1)}(t, r)$ is a solution of the Zerilli equation:

$$\frac{\partial^2 \psi^{(1)}(t, r)}{\partial r^{*2}} - \frac{\partial^2 \psi^{(1)}(t, r)}{\partial t^2} - V(r^*) \psi^{(1)}(t, r) = 0 \quad (10)$$

where

$$r^* = r + 2M \ln[r/(2M) - 1] \quad (11)$$

and

$$V(r) = 6 \left(1 - 2\frac{M}{r}\right) \frac{4r^3 + 4r^2M + 6rM^2 + 3M^3}{r^3(2r + 3M)^2} \quad (12)$$

We remark that, for sufficiently large r , (10) approaches the wave equation, and, therefore, for sufficiently small initial data, $\psi^{(1)}(t, r)$ approaches a function of $t - r^*$, for large r and t .

Going now to the second order in ϵ , $L = 0$, perturbations, we find that there are four non trivial equations (written as $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor), for the second order functions. The equation $R_{tr} = 0$ takes the form

$$\frac{\partial H_2^{(2)}(t, r)}{\partial t} = \mathcal{S} \quad (13)$$

where \mathcal{S} is quadratic in the first order perturbations. We assume a choice of initial data, (for $t = 0$), such that M is equal to the ADM mass, and, therefore, we should have

$$\lim_{r \rightarrow \infty} [r H_2^{(2)}(0, r)] = 0 \quad (14)$$

so that there is no correction to the mass for $t = 0$.

Since on the other hand, for large t , the solution should approach the static black hole configuration, we should have

$$\lim_{t \rightarrow \infty} [H_2^{(2)}(t, r)] = \frac{2\delta M}{r - 2M} \quad (15)$$

and we find that

$$\int_0^\infty \frac{\partial H_2^{(2)}(t, r)}{\partial t} dt = \frac{2\delta M}{r - 2M} - H_2^{(2)}(0, r) \quad (16)$$

and, then,

$$\delta M = \frac{1}{2} \lim_{r \rightarrow \infty} \left[r \int_0^\infty \mathcal{S} dt \right] \quad (17)$$

Using the asymptotic properties of the Zerilli function, and a fair amount of algebra, (some details are given in the Appendix), we find,

$$\delta M = - \lim_{r \rightarrow \infty} \left[\frac{3}{10} \int_0^\infty \left| \frac{\partial \psi}{\partial t} \right|^2 dt \right] \quad (18)$$

which expresses the change in mass in terms of the first order perturbations, through the Zerilli function. But the right hand side of this equation is also immediately recognized as precisely (minus) the total gravitational radiation energy, computed from the Landau - Lifschitz (pseudotensor) equation [4].

3 Comments

The main result of the present analysis, indicated by Eq. (18) is interesting in several respects. To begin with, we remark that it was obtained in the framework of *second order perturbation theory*, namely, the presence of the “source terms”, containing the contributions from the lower (first) order perturbations was crucial in its derivation. Second, we find an expression for the radiated energy, working only in the Regge-Wheeler gauge,

that is totally independent of any consideration of asymptotically flat or radiation gauges and, or identifications of gravitational wave amplitudes. Furthermore, the expression, although computed in second order perturbation theory, contains only information from the first order quantities. Thus, in a sense, it may be considered as a “derivation” of the radiation formula to be used in first order perturbation theory. That this equation coincides with the Landau-Lifschitz pseudotensor prescription, may then be taken as a reassuring (although partial) proof of the physical consistency of the black hole perturbation treatment.

Finally we remark that the proof in this note has been limited to a restricted, although relevant, set of perturbations, since only the first order, $L = 2$ even case was considered. This choice was made mainly for simplicity and we expect that an analogous result should hold in the general case. This will be considered elsewhere.

Appendix

We include in this Appendix some computational details. First we notice that, from $R_{tr} = 0$, after some simplifications using the first order equations, we find

$$\begin{aligned} rH_2^{(2)},t &= \frac{r^2}{10} \left(H_1^{(1)},r H_1^{(1)},t - H_2^{(1)},r H_2^{(1)},t \right) - \frac{r}{5} \left(K^{(1)} H_2^{(1)},t + H_1^{(1)} H_1^{(1)},t - H_2^{(1)} H_2^{(1)},t \right) \\ &\quad - \frac{2}{5} K^{(1)} H_1^{(1)} - \frac{r-2M}{5r} H_1^{(1)} H_2^{(1)} + \frac{M}{5} \left(H_1^{(1)} H_2^{(1)},r - H_1^{(1)},r H_2^{(1)} \right). \end{aligned} \quad (19)$$

Assuming an outgoing wave boundary condition for large r , (appropriate for perturbative initial data), one can show that $\psi(t, r)$ admits the following asymptotic expansion

$$\psi(t, r) = \frac{1}{3} F^{\{3\}} + \frac{1}{r} F^{\{1\}} + \frac{1}{r^2} \left(F - M F^{\{1\}} \right) + \frac{1}{r^3} \left(\frac{7M^2}{4} F^{\{1\}} - M F \right) + \mathcal{O}(1/r^4) \quad (20)$$

where $\mathcal{O}(1/r^4)$ means an expression whose absolute value is bounded by A/r^4 , with A some positive constant, $F = F(t - r^*)$, is an arbitrary real function, and $F^{\{n\}}(x) = d^n F/dx^n$. We assume also that $F(x)$, $F^{\{1\}}(x)$, and $F^{\{2\}}(x)$ are bounded, that $F^{\{3\}}(x)$, is square integrable, and that and

$$\lim_{x \rightarrow +\infty} x^2 F^{\{3\}}(-x) F^{\{4\}}(-x) = 0$$

(These are essentially a “finite energy”, plus “smoothness at infinity” conditions).

Replacing (6), (7), and (8) in (19), and expanding the resulting expression after replacing the asymptotic expression (20) for ψ , we find

$$\begin{aligned} r \frac{\partial H_2^{(2)}}{\partial t} = & -\frac{1}{15}(F^{\{3\}})^2 + \frac{r^2}{45} \frac{\partial}{\partial t} (F^{\{3\}} F^{\{4\}}) + \frac{4r}{45} \frac{\partial}{\partial t} \left[\frac{M}{2} F^{\{3\}} F^{\{4\}} + (F^{\{3\}})^2 \right] \\ & + \frac{\partial}{\partial t} \left[\frac{4M^2}{45} F^{\{3\}} F^{\{4\}} + \frac{5M}{18} (F^{\{3\}})^2 + \frac{2}{15} F^{\{2\}} F^{\{3\}} \right] + \mathcal{O}(1/r) \end{aligned} \quad (21)$$

From the assumptions made above on F , it is clear that the time integrals of the second, third and fourth terms on the right hand side of (21), required to obtain $H_2^{(2)}$ vanish in the limit of large r . We therefore have

$$\lim_{r \rightarrow \infty} \left[r \int_0^\infty \frac{\partial H_2^{(2)}}{\partial t} dt \right] = -\frac{1}{15} \int_0^\infty (F^{\{3\}})^2 dt \quad (22)$$

which leads immediately to (18).

Acknowledgements

I wish to thank R. H. Price, J. Pullin, and C. O. Nicasio for their helpful comments and suggestions. This work was supported in part by the National University of Córdoba, by grant NSF-INT-9512894, and grants from CONICET and CONICOR (Argentina). The author is a member of CONICET.

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Notice that in this paper we use the normalization for ψ given in [5]

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